

# Renormalization of the Volkov propagator

Martin Lavelle and David McMullan

*Centre for Mathematical Sciences, University of Plymouth, Plymouth, PL4 8AA, United Kingdom*



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The perturbative description of an electron propagating in a plane wave background is developed and loop corrections analyzed. The ultraviolet divergences and associated renormalization are studied using the sideband framework within which the multiplicative form of the corrections becomes manifest. An additional renormalization beyond that usually expected is identified and interpreted as a loop correction to the background induced mass term. Results for the strong field sector are conjectured.

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## I. INTRODUCTION

The Volkov solution [1] for an electron in a plane wave background is one of the key theoretical building blocks underpinning our understanding of how matter interacts with a laser. As quantum effects become significant, strong field techniques from quantum electrodynamics (QED) are required. Understanding potential new physics in this high intensity regime is of clear importance and, in turn, should influence plans for future facilities and experiments.

The Volkov solution has been extensively studied over the years and applied to a wide class of problems in both linearly and circularly polarized backgrounds, see e.g., [2–15]. Working in the full elliptic class of polarizations allows for a much clearer description of these systems and helps clarify some of their physical content [16]. In particular, this more general approach shows that the laser induced mass shift is actually independent of the eccentricity of the background.

Loop corrections in a laser background have been looked at several times, as e.g., in [17–22]. Unitarity arguments are often used to directly link loop corrections to effective cross sections. It has been argued, see e.g., [23], that the laser background has no impact on the renormalization of the theory. To have confidence in this result, it is important to probe the loop structures and associated renormalization of the theory in a variety of ways. In this paper we will do this by taking a weak field perspective which has the advantage that standard perturbative techniques can be directly applied.

The propagation of an electron in a laser background is often denoted by a double line. This notation represents the inclusion of multiple interactions with the laser. A physical

way to think of this is that the double line incorporates all degenerate processes, i.e., the emission and absorption of photons indistinguishable from the background. This is reminiscent of the Lee-Nauenberg approach to the infrared problem [24], see also [25].

We take the double line to mean the two point function for the Volkov field in the plane wave background, see Fig. 1. We will, in this paper, make this link precise in terms of emission into and absorption from the laser. Throughout this paper, we will distinguish between absorption (dashed lines coming in from the left) and emission (dashed lines going out to the right). The incorporation of loops in the weak field limit will then follow using standard perturbative methods. This will then allow us to better understand the way in which loops are added to the double line, see Fig. 2. As we shall see, clarifying the links between these descriptions of matter propagating in a laser will reveal important points about the renormalization of such charges.

In this paper we will study the renormalization of the theory describing an electron propagating in a plane wave background. This analysis will start in the weak intensity regime and there we will calculate the ultraviolet divergences

$$\text{Double line } p = \sum_{\text{laser interactions}} \text{Diagram with } p_{\text{in}}, p_{\text{out}} \text{ and laser interactions}$$

FIG. 1. Double line representation of an electron propagating in a background at tree level.

$$\text{Double line with loop} = \sum_{\text{laser interactions and loops}} \text{Diagram with } p_{\text{in}}, p_{\text{out}} \text{ and laser interactions and loops}$$

FIG. 2. One-loop correction to the double line representation.

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that arise at one loop. Our loop calculations will be in the Feynman gauge. Polarization effects will be clarified through working with the full elliptic class at all times. As well as the naively expected ultraviolet structures, which are independent of the background, we will identify an additional correction to the laser induced mass shift. We show through explicit calculations of higher order laser interactions that they are renormalized by the same additional correction, and conjecture that this is universal for this class of backgrounds. Renormalization is most easily studied within a momentum space description of the theory, so we conclude with a discussion on how a consistent momentum space language can be applied to this system where translation invariance has been broken by the background field, and conjecture all orders results.

## II. THE PERTURBATIVE SETUP

An electron propagating through a laser can absorb multiple photons from the background. Additionally, such an electron can emit photons which are degenerate with, and indistinguishable from, the background. Both types of interactions are, as we shall discuss, required for the double line description. If, however, the electron emits a photon that is distinguishable from the background then this corresponds to nonlinear Compton scattering rather than propagation.

We will argue that summing in a suitable way over all such degeneracies leads to the Volkov description [1] of an electron propagating through such a background. What is more, this will allow for a direct route to the incorporation of loop corrections in such processes and hence the renormalization of the theory.

The momentum of an electron in a plane wave background can be decomposed into some initial momentum  $p$ , along with multiples of the null momentum  $k$  characterizing the background. We denote by  $P_n$  the resulting propagator after  $n$  net laser absorptions:

$$P_n = \frac{i}{\not{p} + n\not{k} - m + i\epsilon}. \quad (1)$$

Note that in terms of the overall momentum for the electron, we view an emission as a negative absorption from the laser. So if there were two absorptions and one emission, say, then  $n = 1$ . This compact notation for the propagators will provide the building blocks for our description of both tree and loop corrected propagation.

For example, an additional absorption by the electron is described by the incoming interaction shown in Fig. 3, where the absorption factor  $A$  between the propagators is given by

$$A = -i\mathcal{A}. \quad (2)$$

Here  $\mathcal{A}_\mu$  is essentially the coupling,  $e$ , times the Fourier component of the classical potential for an elliptically

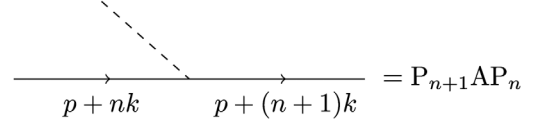


FIG. 3. Single absorption from the background.

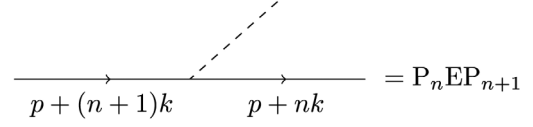


FIG. 4. Single emission into the background.

polarized plane wave. We will expand on this terminology later, but see also [16] for more details on this formalism and the connection to the Stokes' parameter description of the background.

The associated emission of a photon degenerate with the background is described by the outgoing process given in Fig. 4, where the emission factor  $E$  is given by

$$E = -i\mathcal{A}^*. \quad (3)$$

It is helpful here to clarify the notation being used. By  $\mathcal{A}^*$  we mean the slashed version of the conjugated field, so  $\mathcal{A}^* = \mathcal{A}_\mu^* \gamma^\mu$ . This is a useful shorthand for the unambiguous expression for the dual field,

$$\bar{\mathcal{A}} := \gamma_0 \mathcal{A}^\dagger \gamma_0 \equiv \mathcal{A}_\mu^* \gamma^\mu. \quad (4)$$

Note that acting on the propagators we have the duality relation  $\bar{P}_n = -P_n$  and on the absorption term  $\bar{A} = -E$ . The duality transformation needs to respect the time ordering implicit in the  $i\epsilon$  prescription. This means that formally we should take  $\bar{\epsilon} = -\epsilon$ . Overall, the processes in Figs. 3 and 4 are (anti)dual to each other in the sense that

$$\overline{P_{n+1} A P_n} = -P_n E P_{n+1}. \quad (5)$$

We now turn to the one-loop corrections to the basic interactions between the matter and its background. For the absorption process in Fig. 3 we have, at one loop, the three diagrams in Fig. 5, and, for the emission process of Fig. 4, we get the contributions in Fig. 6. Note that the central term for each row here has the structure of a vertex correction, while the other terms are self-energies for the external legs. So it is not immediately clear that grouping them together

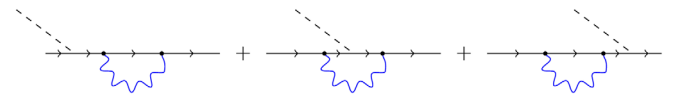


FIG. 5. Single absorption with a loop correction.

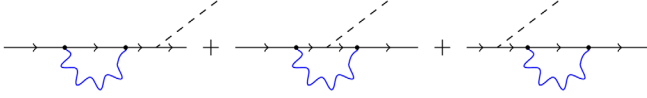


FIG. 6. Single emission with a loop correction.

in this way leads to a multiplicative renormalization of the tree level processes in Figs. 3 and 4.

To clarify how renormalization works in this context, we first need to recall how sideband structures emerge from the tree level diagrams in Figs. 3 and 4. To that end, we note that the absorption process of Fig. 3 can be written as the difference of two propagators:

$$P_{n+1}AP_n = IP_n - P_{n+1}I, \quad (6)$$

where we define the “In” term as

$$I = \frac{2p \cdot \mathcal{A} + \not{k}\mathcal{A}}{2p \cdot k}. \quad (7)$$

The simple identity (6) lies at the heart of the sideband description of this propagation that was introduced in [2]. At its heart, it is simply a partial fraction expansion which relates the products of propagators to their sums.

The corresponding emission version of this sideband identity can be easily deduced by using the duality transformation (5) and (6) to give

$$P_n EP_{n+1} = -\overline{P_{n+1}AP_n} = P_n O - OP_{n+1}, \quad (8)$$

where the “Out” insertion is given by

$$O := \bar{I} = \frac{2p \cdot \mathcal{A}^* - \not{k}\mathcal{A}^*}{2p \cdot k}. \quad (9)$$

The matrix nature of the I and O terms means that we must be careful with the ordering in (6) and (8). However, due to the null nature of  $\not{k}$  and the fact that it commutes with both  $\mathcal{A}$  and  $\mathcal{A}^*$ , we find that the In and Out terms commute:

$$[I, O] = 0. \quad (10)$$

Having clarified the tree level structures in Figs. 3 and 4, we can now analyze in much the same way the loop corrections of Figs. 5 and 6.

The ultraviolet poles related to the self-energy contributions of Fig. 5 can be readily calculated by using standard results from QED, see e.g., chapter 18 of [26]. Working in the Feynman gauge, and using dimensional regularization in  $D = 4 - 2\epsilon$  dimensions, we have for incoming momentum  $p + nk$  the contribution described in Fig. 7. After some simplifications of the gamma matrices, we have for the ultraviolet divergent structure

$$\text{Diagram} = P_n \Sigma_n P_n$$

FIG. 7. One-loop self-energy correction to the propagator.

$$\Sigma_n = -e^2 \mu^{2\epsilon} \int_{\text{UV}} \frac{d^D s}{(2\pi)^D} \frac{(2-D)\not{s} + Dm}{(s - (p + nk))^2 (s^2 - m^2)}, \quad (11)$$

where  $s$  is the four-momentum of the electron in the loop so that the photon in the loop has four-momentum  $p + nk - s$ .

Retaining only the ultraviolet pole gives

$$\Sigma_n = (i3m + P_n^{-1})\delta_{\text{UV}}. \quad (12)$$

The notation here is that, from (1),  $P_n^{-1} = (-i)(\not{p} + n\not{k} - m)$  while the ultraviolet pole is given by

$$\delta_{\text{UV}} = -\frac{e^2}{(4\pi)^2} \frac{1}{\epsilon}. \quad (13)$$

Substituting the self-energy expression (12) into Fig. 7 gives the familiar double pole mass term and a single pole. So the first and third diagrams in Fig. 5 become

$$\begin{aligned} & P_{n+1}\Sigma_{n+1}P_{n+1}AP_n + P_{n+1}AP_n\Sigma_nP_n \\ &= IP_n\Sigma_nP_n - P_{n+1}\Sigma_{n+1}P_{n+1}I + P_{n+1}\Sigma_{n+1}IP_n - P_{n+1}I\Sigma_nP_n. \end{aligned} \quad (14)$$

The first two terms on the right-hand side here are the sideband structures but the final two include a mixture of momenta.

The vertex correction term in Fig. 5 is still to be included. The corresponding Feynman rule for this is given in Fig. 8. From this we have

$$\begin{aligned} \Sigma_{\text{in}} &= -e^2 \mu^{2\epsilon} \int_{\text{UV}} \frac{d^D s}{(2\pi)^D} \frac{\gamma^\rho (\not{s} + \not{k} + m) \mathcal{A} (\not{s} + m) \gamma^\tau}{((s + k)^2 - m^2)(s^2 - m^2)} \\ &\quad \times \frac{g_{\rho\tau}}{(s - (p + nk))^2}. \end{aligned} \quad (15)$$

Retaining only the ultraviolet divergent structures, which can easily be recognized by power counting, we find

$$\Sigma_{\text{in}} = i\mathcal{A}\delta_{\text{UV}} = -A\delta_{\text{UV}}. \quad (16)$$

Here we recognize the tree level absorption factor of Fig. 3 multiplied by the above ultraviolet pole. We emphasise

$$\text{Diagram} = P_{n+1}\Sigma_{\text{in}}P_n$$

FIG. 8. The one-loop absorption vertex correction to the propagator.

that, in the last step, the  $e^2$  factor from the loop is in the  $\delta_{UV}$  term, while the background coupling factor of  $e$  has been absorbed into the definitions of  $\tilde{A}$  and  $A$ .

This simple relation for the ultraviolet structure of this vertex term means that we can exploit the sideband relation (6) to rewrite

$$\begin{aligned}\Sigma_{in} &= I(\delta_{UV}P_n^{-1}) - (\delta_{UV}P_{n+1}^{-1})I, \\ &= I\Sigma_n - \Sigma_{n+1}I.\end{aligned}\quad (17)$$

Note that the scalar mass terms canceled in the last step. The second diagram in Fig. 5 can thus be written as

$$P_{n+1}\Sigma_{in}P_n = P_{n+1}I\Sigma_nP_n - P_{n+1}\Sigma_{n+1}IP_n. \quad (18)$$

We can now write the sum of the three diagrams in Fig. 5 as

$$\begin{aligned}P_{n+1}\Sigma_{n+1}P_{n+1}AP_n + P_{n+1}\Sigma_{in}P_n + P_{n+1}AP_n\Sigma_nP_n \\ = IP_n\Sigma_nP_n - P_{n+1}\Sigma_{n+1}P_{n+1}I.\end{aligned}\quad (19)$$

Note that all of the nonsideband structures cancel. What remains has exactly the same structure as the sideband description of the tree level result (6), but with the expected self-energy corrections to the sideband propagators. We thus see the attractive result that the loop corrections to (6) generate the normal one-loop propagator corrections to the tree level propagators in the sidebands:

$$I(P_n + P_n\Sigma_nP_n) - (P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})I. \quad (20)$$

The interpretation of this result is then direct: it will lead to the sidebands requiring the standard mass and wave function renormalizations.

The emission process of Fig. 4, and its loop corrections in Fig. 6, then lead to the sidebands described in (8) being renormalized in a similar way. The key out-going vertex identity, dual to (18), is that

$$P_n\Sigma_{out}P_{n+1} = P_n\Sigma_nOP_{n+1} - P_nO\Sigma_{n+1}P_{n+1}, \quad (21)$$

where we have used  $\tilde{\Sigma}_{in} = -\Sigma_{out}$  and  $\tilde{\Sigma}_n = -\Sigma_n$ . This then results in the loop corrections to the sidebands (8) being given by

$$(P_n + P_n\Sigma_nP_n)O - O(P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1}). \quad (22)$$

This we see is precisely (minus) the dual of (20) as we would naively expect from the tree level relation (5). Again, the standard mass and wave-function renormalizations will suffice.

### III. HIGHER ORDER BACKGROUND CORRECTIONS

Having understood the structure of the loop correction to a single absorption or emission of a laser photon by the electron, we now want to calculate the ultraviolet divergences when multiple photons are absorbed or emitted. We shall consider the case of both absorption and emission in the following section.

The first thing to note is that loops spanning more than one laser absorption or emission, as depicted in Fig. 9, are all finite in the ultraviolet regime by simple power counting.

This means that when, e.g., we consider the tree level double absorption process, where the incoming propagator  $P_n$  absorbs two additional laser photons, as in Fig. 10, then we need only to consider the loop corrections straddling no more than one background vertex, as shown in Fig. 11. In this we again see a mixture of self-energy and single vertex corrections.

In order to understand and interpret these corrections, we need to first identify the sideband structures in the tree level term shown in Fig. 10. To that end, we write this as

$$P_{n+2}AP_{n+1}AP_n = P_{n+2}AP_{n+1}P_{n+1}^{-1}P_{n+1}AP_n. \quad (23)$$

This allows us to use the absorption relation (6) twice, resulting in four terms:

$$P_{n+2}AP_{n+1}AP_n = P_{n+2}I^2 - IP_{n+1}I + I^2P_n - P_{n+2}IP_{n+1}^{-1}IP_n. \quad (24)$$

A key identity needed here, which is straightforward to show, is that

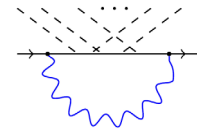


FIG. 9. Loop spanning multiple laser interactions.

$$\text{Diagram} = P_{n+2}AP_{n+1}AP_n$$

FIG. 10. Tree level double absorption process.

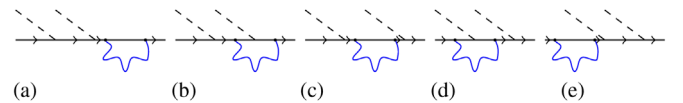


FIG. 11. Double absorption process with a loop correction.

$$\mathbb{I}P_{n+1}^{-1}\mathbb{I} = \left(\frac{1}{2}\mathbb{I}^2 + \frac{1}{2}v\right)P_n^{-1} + P_{n+2}^{-1}\left(\frac{1}{2}\mathbb{I}^2 - \frac{1}{2}v\right), \quad (25)$$

where

$$v := \frac{\mathcal{A} \cdot \mathcal{A}}{2p \cdot k}. \quad (26)$$

Using this identity in (24), we see that the sidebands for the double absorption process depicted in Fig. 10 are given by

$$\begin{aligned} P_{n+2}AP_{n+1}AP_n &= \left(\frac{1}{2}\mathbb{I}^2 - \frac{1}{2}v\right)P_n - \mathbb{I}P_{n+1}\mathbb{I} \\ &+ P_{n+2}\left(\frac{1}{2}\mathbb{I}^2 + \frac{1}{2}v\right). \end{aligned} \quad (27)$$

We will now show that the one-loop diagrams in Fig. 11 generate the expected, ultraviolet one-loop corrections to these three sidebands.

The loop correction can be evaluated by recognizing in the diagrams of Fig. 11 a connection to the earlier loop processes evaluated in the previous section. The first three diagrams represent an initial absorption process followed by the loop corrections of Fig. 5, with shifted initial momentum. In a similar way, the final three diagrams in Fig. 11 can be interpreted as the loop corrections of Fig. 5, followed immediately by an absorption process. These two simplifications double count the middle process, Fig. 11c, so this needs to be subtracted from the combined sum.

Following this reduction prescription, the diagrams in Fig. 11 can then be evaluated using the loop results (20) and the sideband identity (6). This results in terms containing combinations of the form  $\mathbb{I}\Sigma_n\mathbb{I}$  which, from the self-energy extension to (25), can be evaluated by using the identity

$$\mathbb{I}\Sigma_{n+1}\mathbb{I} = \left(\frac{1}{2}\mathbb{I}^2 + \frac{1}{2}v\right)\Sigma_n + \Sigma_{n+2}\left(\frac{1}{2}\mathbb{I}^2 - \frac{1}{2}v\right). \quad (28)$$

From this we rapidly arrive at the sideband structure of the one-loop corrections of Fig. 11 to the double absorption process shown in Fig. 10. Combined with the tree level result, this yields

$$\begin{aligned} &\left(\frac{1}{2}\mathbb{I}^2 - \frac{1}{2}v\right)(P_n + P_n\Sigma_nP_n) - \mathbb{I}(P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})\mathbb{I} \\ &+ (P_{n+2} + P_{n+2}\Sigma_{n+2}P_{n+2})\left(\frac{1}{2}\mathbb{I}^2 + \frac{1}{2}v\right). \end{aligned} \quad (29)$$

Again we see that the sidebands pick up the expected loop corrections.

The double emission process can be evaluated in a similar fashion, or more directly by taking the dual of the double absorption process. The result is that the loop corrections to the double emission processes described in Fig. 12 are given by the sideband terms:

$$= P_{n-2}EP_{n-1}EP_n$$

FIG. 12. Tree level double emission process.

$$\begin{aligned} &(P_n + P_n\Sigma_nP_n)\left(\frac{1}{2}\mathbb{O}^2 - \frac{1}{2}v^*\right) - \mathbb{O}(P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})\mathbb{O} \\ &+ \left(\frac{1}{2}\mathbb{O}^2 + \frac{1}{2}v^*\right)(P_{n+2} + P_{n+2}\Sigma_{n+2}P_{n+2}), \end{aligned} \quad (30)$$

where now

$$v^* := \frac{\mathcal{A}^* \cdot \mathcal{A}^*}{2p \cdot k}. \quad (31)$$

An important point to note here is that the terms  $v$  and  $v^*$  induced by the background do not acquire loop corrections and hence are not renormalized at one loop. We also note that both  $v$  and  $v^*$  are polarization dependent and vanish for a circularly polarized laser, see [16].

#### IV. ABSORPTION AND EMISSION FROM THE BACKGROUND

It is well known that the laser induced mass shift is only generated by processes where there is both emission and absorption from the laser. This is understood at all orders in the background field and is known to be polarization independent, see [16] and references therein. Here we will calculate the one-loop corrections to this important process and see the necessity for a new renormalization.

There are two contributions to the mixed absorption and emission process at the lowest order in the background interactions, as summarized in Fig. 13. We expect from [14] that these diagrams will generate three sidebands and the central one will involve a double pole corresponding to the laser induced mass shift.

If the incoming momentum is again  $p + nk$ , then these diagrams are given by

$$P_nEP_{n+1}AP_n + P_nAP_{n-1}EP_n. \quad (32)$$

Note that each of these contributions is unchanged (up to a sign) by the duality transformations introduced earlier.

The terms in (32) can be evaluated by again inserting appropriate inverse propagators so that both the absorption and emission identities, (6) and (8), can be used. From this we quickly find that

$$+$$

FIG. 13. Tree level absorption and emission corrections.



$$\begin{aligned}
& P_n E P_{n+1} A P_n + P_n A P_{n-1} E P_n \\
& = I P_{n-1} O - 2 O I P_n - P_n 2 O I + O P_{n+1} I \\
& \quad + P_n (O P_{n+1}^{-1} I + I P_{n-1}^{-1} O) P_n.
\end{aligned} \quad (33)$$

The first four terms in this involve the expected sidebands for these processes, but the coefficients are not as expected. The final term needs more work to be interpreted, but should correct these coefficients.

The structure in the brackets in the last equation is analogous to the double absorption contribution seen earlier in (25). The key identity now is that

$$O P_{n+1}^{-1} I + I P_{n-1}^{-1} O = O I P_n^{-1} + P_n^{-1} O I - i \mathcal{M}, \quad (34)$$

where we define the important quantity

$$\mathcal{M}_\mu := -\frac{\mathcal{A}^* \cdot \mathcal{A}}{p \cdot k} k_\mu. \quad (35)$$

Note that  $\mathcal{M} = \bar{\mathcal{M}}$ .

Using (34) we find that the sidebands for this process are given at this order by

$$\begin{aligned}
& P_n E P_{n+1} A P_n + P_n A P_{n-1} E P_n \\
& = I P_{n-1} O - O I P_n - P_n O I - P_n i \mathcal{M} P_n + O P_{n+1} I.
\end{aligned} \quad (36)$$

Here we recognize the expected three sidebands,  $P_n$ ,  $P_{n\pm 1}$ , and the double pole. These terms must be interpreted as corrections, induced by the laser, to the free propagator in the central sideband,  $P_n$ .

Since  $\mathcal{M}$  is in the double pole term for the central sideband, we can relate it to the more familiar polarization independent effective mass,  $m_*$ , induced by the background. Following the discussion in [16], we can write at this order in the laser background,  $P_n - P_n i \mathcal{M} P_n$  as

$$\begin{aligned}
& \frac{i}{\not{p} + n\!\!\!\not{k} - m + i\epsilon} + \frac{1}{\not{p} + n\!\!\!\not{k} - m + i\epsilon} \mathcal{M} \frac{i}{\not{p} + n\!\!\!\not{k} - m + i\epsilon} \\
& \approx \frac{i}{\not{p} + n\!\!\!\not{k} - (m + \mathcal{M}) + i\epsilon}, \\
& = \frac{i(\not{p} + n\!\!\!\not{k} + m - \mathcal{M})}{(p + nk)^2 - m_*^2 + i\epsilon},
\end{aligned} \quad (37)$$

where

$$m_*^2 = m^2 + \not{p} \mathcal{M} + \mathcal{M} \not{p} = m^2 - 2\mathcal{A}^* \cdot \mathcal{A}. \quad (38)$$

Note that the last result is often rewritten as  $m_*^2 = m^2 - e^2 a^2$  where  $-a^2 > 0$  is the amplitude squared of the background.

We now want to calculate the ultraviolet loop corrections to these processes. They are given by the five diagrams in Fig. 14 and the corresponding terms in Fig. 15. Again we

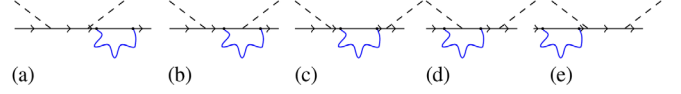


FIG. 14. Absorption then emission with a loop correction.

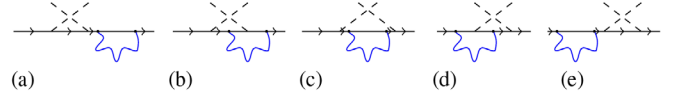


FIG. 15. Emission then absorption with a loop correction.

stress that since we are only calculating the ultraviolet divergences, loops straddling two laser lines may be ignored.

The strategy for evaluating these diagrams mirrors that seen before: we can identify subterms that have already been evaluated, then use the loop generalization of the identity (34), which is

$$O \Sigma_{n+1} I + I \Sigma_{n-1} O = O I \Sigma_n + \Sigma_n O I + i \mathcal{Z}_\mathcal{M}, \quad (39)$$

where we have defined the loop correction to the final term in (34):

$$\mathcal{Z}_\mathcal{M} = \frac{e^2}{(4\pi)^2} \frac{1}{\epsilon} \mathcal{M}. \quad (40)$$

From this we find that the loop corrections to the central sidebands (36) are given by (ignoring higher order terms in the coupling)

$$\begin{aligned}
& I(P_{n-1} + P_{n-1} \Sigma_{n-1} P_{n-1}) O \\
& - O I (P_n + P_n \Sigma_n P_n) - (P_n + P_n \Sigma_n P_n) O I \\
& - (P_n + P_n \Sigma_n P_n) i(\mathcal{M} + \mathcal{Z}_\mathcal{M}) (P_n + P_n \Sigma_n P_n) \\
& + O(P_{n+1} + P_{n+1} \Sigma_{n+1} P_{n+1}) I.
\end{aligned} \quad (41)$$

We have written it in this way to bring out the multiplicative structure of these corrections at this order. Terms without a  $\Sigma$  are tree level, terms with one  $\Sigma$  are our one-loop results, while terms with products of two or more  $\Sigma$  factors remain to be verified in further work as the calculations reported here are only to one loop.

Written in this way, we see a new structure in the third line of the loop corrections:  $P_n(-i \mathcal{Z}_\mathcal{M}) P_n$ . This, as we will discuss in more detail later, corresponds to a new renormalization being needed in this theory. This will be a renormalization of the laser induced mass shift,  $\mathcal{M}$ .

This last result is unexpected and requires further testing. To do this we now consider a process that is higher order in the background interaction and that also generates a laser induced mass shift at tree level. To be concrete, we will consider two absorptions and one emission.

This is an interesting process as the mixture of absorptions and an emission will induce both  $\mathcal{A}^* \cdot \mathcal{A}$  and  $\mathcal{A} \cdot \mathcal{A}$



FIG. 16. Two absorptions and one emission at tree level.

terms, and it is not *a priori* clear if there will be interference between loop corrections. The tree level process of interest in this respect is thus given by the three processes in Fig. 16.

We now expect four sidebands with propagators  $P_{n+2}$ ,  $P_{n+1}$ ,  $P_n$  and  $P_{n-1}$ . There will also be a mixture of the  $v$  term seen in Fig. 10 and the mass term found in the central

band of Fig. 13. The ultraviolet loop corrections will now generate 21 graphs. The strategy for evaluating these is again to group terms so that we get a mixture of previously evaluated subterms and absorption or emission factors analogous to (28) and (34). The end result of this gives the loop corrections summarized within the full (tree level and loop) sideband structures:

$$\begin{aligned} & \left(\frac{1}{2}I^2 - \frac{1}{2}v\right)(P_{n-1} + P_{n-1}\Sigma_{n-1}P_{n-1})O - O\left(\frac{1}{2}I^2 - \frac{1}{2}v\right)(P_n + P_n\Sigma_nP_n) - I(P_n + P_n\Sigma_nP_n)OI \\ & - I(P_n + P_n\Sigma_nP_n)i(\mathcal{M} + \mathcal{Z}_{\mathcal{M}})(P_n + P_n\Sigma_nP_n) + (P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})O\left(\frac{1}{2}I^2 + \frac{1}{2}v\right) \\ & + OI(P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})I + (P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})i(\mathcal{M} + \mathcal{Z}_{\mathcal{M}})(P_{n+1} + P_{n+1}\Sigma_{n+1}P_{n+1})I \\ & - O(P_{n+2} + P_{n+2}\Sigma_{n+2}P_{n+2})\left(\frac{1}{2}I^2 + \frac{1}{2}v\right). \end{aligned} \quad (42)$$

Here we see clearly the same structures in these loop corrections as encountered earlier in (20), (22), (29), (30) and (41). From this result we can immediately deduce the corresponding dual process involving two emissions and just one absorption from the background. We see that there is no interference between the mass terms and the  $v^*$  insertions.

To summarize, these detailed perturbative investigations show that the loop corrections to the propagation of an electron in a plane wave background preserve the sideband structures and, through that, induce the expected one-loop corrections to the normalization of the propagators, including the vacuum mass shift. Unexpectedly, we have seen that the laser induced mass also has an ultraviolet correction. Having exposed and isolated these loop structures, we now address the (minimal) renormalization needed for the extraction of finite, physical results.

## V. RENORMALIZATION

We have seen, through multiple examples, that the sideband structure of the theory is preserved when loop corrections are included. To understand the renormalization of the theory, let us consider the  $\ell^{\text{th}}$  sideband. For this sideband we have seen that the loop corrections induce a replacement,

$$P_\ell \rightarrow P_\ell + P_\ell \Sigma_\ell P_\ell, \quad (43)$$

together with an additional correction to the background induced, mass shift

$$-i\mathcal{M} \rightarrow -i(\mathcal{M} + \mathcal{Z}_{\mathcal{M}}). \quad (44)$$

The ultraviolet divergences in  $\Sigma_\ell$  and  $\mathcal{Z}_{\mathcal{M}}$ , see (12) and (40), signal the need for renormalization. This we now introduce by shifting from bare to renormalized quantities.

It is useful here to refine our notation, introduced in Eq. (1), for the sideband propagator to include both normal and induced mass terms, as in (37). We thus define

$$P_\ell(m, \mathcal{M}) := \frac{i}{\not{p} + \not{\ell} \not{k} - (m + \mathcal{M}) + i\epsilon}. \quad (45)$$

Then the loop corrections, (43) and (44), can be written more succinctly as

$$P_\ell(m, \mathcal{M}) \rightarrow P_\ell(m, \mathcal{M}) + P_\ell(m, \mathcal{M})(\Sigma_\ell - i\mathcal{Z}_{\mathcal{M}})P_\ell(m, \mathcal{M}). \quad (46)$$

To now renormalize this sector of the theory, we follow the usual prescription whereby we first interpret these results as arising from working with the bare Volkov fields and masses:  $\psi_V^B$ ,  $m^B$  and  $\mathcal{M}_\mu^B$ . Then we define the physical, renormalized quantities  $\psi_V$ ,  $m$  and  $\mathcal{M}_\mu$  by

$$\psi_V^B := \mu^{-\epsilon} \sqrt{Z_2} \psi_V = \mu^{-\epsilon} \sqrt{1 + \delta_2} \psi_V, \quad (47)$$

$$m^B := Z_m m = (1 + \delta_m) m \quad (48)$$

and

$$\mathcal{M}_\mu^B := Z_{\mathcal{M}} \mathcal{M}_\mu = (1 + \delta_{\mathcal{M}}) \mathcal{M}_\mu. \quad (49)$$

These counterterms are then determined by the requirement that when we work with renormalized quantities, we obtain finite results. Note that the mass scale  $\mu^{-\epsilon}$  in the wave function renormalization factor can be neglected in the leading order analysis presented here.

The full, renormalized sideband propagator at this order is then, from (46),

$$Z_2^{-1} P_\ell(m^B, \mathcal{M}^B) + P_\ell(m, \mathcal{M})(\Sigma_\ell - i\mathfrak{Z}_{\mathcal{M}}) P_\ell(m, \mathcal{M}). \quad (50)$$

In the second term of this expression the presence of the loop corrections means that renormalized quantities can be immediately used when working with the leading order loop corrections. In the first term, though, we are still explicitly working with the bare fields.

These bare quantities can be expanded to give

$$\begin{aligned} Z_2^{-1} P_\ell(m^B, \mathcal{M}^B) &= (1 - \delta_2) P_\ell((1 + \delta_m)m, (1 + \delta_{\mathcal{M}})\mathcal{M}) \\ &= P_\ell(m, \mathcal{M}) + P_\ell(m, \mathcal{M}) \\ &\quad \times (-P_\ell^{-1} \delta_2 - i(m\delta_m + \mathcal{M}\delta_{\mathcal{M}})) P_\ell(m, \mathcal{M}). \end{aligned} \quad (51)$$

Thus the renormalized sideband propagator, (50), becomes

$$P_\ell(m, \mathcal{M}) + P_\ell(m, \mathcal{M}) \Sigma_\ell^R P_\ell(m, \mathcal{M}), \quad (52)$$

where

$$\Sigma_\ell^R = \Sigma_\ell - P_\ell^{-1} \delta_2 - i(m\delta_m + \mathfrak{Z}_{\mathcal{M}} + \mathcal{M}\delta_{\mathcal{M}}). \quad (53)$$

From this, and Eqs. (12) and (40), we see that, independent of the sideband being considered, the minimal renormalization prescription corresponds to the familiar results that

$$\delta_2 = \delta_{UV} \quad \text{and} \quad \delta_m = 3\delta_{UV}, \quad (54)$$

along with the additional requirement that

$$\delta_{\mathcal{M}} = \delta_{UV}, \quad (55)$$

where  $\delta_{UV}$  was defined in Eq. (13). Our higher order calculations, in terms of absorptions and emissions, support the expectation that the renormalization prescriptions (54) and (55) hold also in the strong field sector. We will now recall how the sideband formulation can be extended to all such orders and, through this, conjecture the form of the one-loop corrections to the full Volkov description of an electron propagating through a plane wave, laser background.

## VI. THE FULL VOLKOV DESCRIPTION AT ONE LOOP

The importance of the Volkov solution for the tree level results is that the sideband description, discussed above in the perturbative framework, is known to all orders in the background interaction for this wide class of polarizations, see [16]. We now want to develop the link between the perturbative loop structures presented here and that all orders formalism. In doing so we shall see that the perturbative results actually motivate a significant simplification to the all orders description. Armed with that result, we shall be able to conjecture a compact expression for the leading one-loop corrections at all orders in the intensity of the background.

The exact tree level solution for the two point function describing an electron propagating in an elliptically polarized background can be written [see Eq. (44) in [16] and discussions therein] as the usual momentum space integration factors times the double sum over  $r$  and  $s$  of the following sideband structures:

$$\begin{aligned} e^{irk \cdot x} &\left( J_{s+r}^\tau(p) + \frac{k \cdot \mathcal{A}}{2p \cdot k} J_{s+r+1}^\tau(p) + \frac{k \cdot \mathcal{A}^*}{2p \cdot k} J_{s+r-1}^\tau(p) \right) \\ &\times P_s(m, \mathcal{M}) \left( J_s^{\tau*}(p) - \frac{k \cdot \mathcal{A}^*}{2p \cdot k} J_{s+1}^{\tau*}(p) - \frac{k \cdot \mathcal{A}}{2p \cdot k} J_{s-1}^{\tau*}(p) \right). \end{aligned} \quad (56)$$

Unpicking (56) we see that, as we sum over  $s$ , the sideband propagator  $P_s(m, \mathcal{M})$  is sandwiched between factors built out of (generalized) Bessel functions,  $J_\ell^\tau(p)$ , where the parameter  $\ell$  can be various combinations of the summation parameters  $r$  and  $s$ . These Bessel functions are also labeled by the eccentricity parameter  $\tau$  characterizing the polarization of the background in the elliptic class. The precise definition of these Bessel functions is that

$$\begin{aligned} J_\ell^\tau(p) &:= J_\ell(\omega_1, v, \omega_2) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i(\omega_1 \sin \theta + v \sin 2\theta + \omega_2 \cos \theta)} e^{-i\ell\theta}, \end{aligned} \quad (57)$$

where the eccentricity information is now encoded in the real parameters  $\omega_1$ ,  $v$  and  $\omega_2$ . The connection with the complex vector parameters  $\mathcal{A}$  and  $\mathcal{A}^*$ , introduced in (2) and (3), is seen in Eq. (26) for  $v$ , and the definitions

$$\begin{aligned} \omega_1 &= -\left( \frac{p \cdot \mathcal{A}}{p \cdot k} + \frac{p \cdot \mathcal{A}^*}{p \cdot k} \right) \quad \text{and} \\ \omega_2 &= -i \left( \frac{p \cdot \mathcal{A}}{p \cdot k} - \frac{p \cdot \mathcal{A}^*}{p \cdot k} \right). \end{aligned} \quad (58)$$

The fact that  $v$  is real is perhaps surprising and seems at odds with our effort, as in (31), to distinguish typographically between  $v$  and  $v^*$ . However, this was a useful



bookkeeping device to keep track of the duality structures seen earlier, and one that we will return to below.

From the perturbative perspective, one of the most striking and immediate things to note about the all orders result (56) is the absence of the variables that were the building blocks in the description developed in this paper. In particular, the In and Out terms, (7) and (9), seem to be absent.

Give the central role played by these terms in our perturbative analysis, it seems logical to try to rewrite the all orders result in terms of them. To this end, we make the change of variables  $\omega_1 \rightarrow \Omega_1$  and  $\omega_2 \rightarrow \Omega_2$ , with

$$\Omega_1 = \omega_1 - \frac{k\mathcal{A} - k\mathcal{A}^*}{2p \cdot k} = -(I + O) \quad (59)$$

and

$$\Omega_2 = \omega_2 - i \frac{k\mathcal{A} + k\mathcal{A}^*}{2p \cdot k} = -i(I - O). \quad (60)$$

Note that the reality requirements on  $\omega_1$  and  $\omega_2$  are now replaced by the duality result that  $\bar{\Omega}_1 = \Omega_1$  and  $\bar{\Omega}_2 = \Omega_2$ . The trivial commutativity of  $\omega_1$  and  $\omega_2$  is now the non-trivial matrix result that  $\Omega_1 \Omega_2 = \Omega_2 \Omega_1$ , which is ensured by the null properties of the background field.

These commutativity and duality relations enable us to extend the domain of the Bessel functions defined in (57) so that we can unambiguously write

$$J_\ell(\Omega_1, v, \Omega_2) := \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i(\Omega_1 \sin \theta + v \sin 2\theta + \Omega_2 \cos \theta)} e^{-i\ell\theta}. \quad (61)$$

To understand the connection between these extended functions and the complicated pre- and postfactors in the two point function (56), we note that the null property of the vector  $k$  means that

$$e^{i\Omega_1 \sin \theta} = e^{i\omega_1 \sin \theta} \left( 1 - i \frac{k\mathcal{A} - k\mathcal{A}^*}{2p \cdot k} \sin \theta \right) \quad (62)$$

and

$$e^{i\Omega_2 \cos \theta} = e^{i\omega_2 \cos \theta} \left( 1 + \frac{k\mathcal{A} + k\mathcal{A}^*}{2p \cdot k} \cos \theta \right). \quad (63)$$

Hence we quickly see that

$$J_\ell(\Omega_1, v, \Omega_2) = J_\ell(\omega_1, v, \omega_2) + J_{\ell+1}(\omega_1, v, \omega_2) \frac{k\mathcal{A}}{2p \cdot k} + J_{\ell-1}(\omega_1, v, \omega_2) \frac{k\mathcal{A}^*}{2p \cdot k} \quad (64)$$

and

$$\bar{J}_\ell(\Omega_1, v, \Omega_2) = J_\ell^*(\omega_1, v, \omega_2) - J_{\ell+1}^*(\omega_1, v, \omega_2) \frac{k\mathcal{A}^*}{2p \cdot k} - J_{\ell-1}^*(\omega_1, v, \omega_2) \frac{k\mathcal{A}}{2p \cdot k}. \quad (65)$$

We can thus write the two point function (56) in a much more compact way as the sum of all terms of the form

$$e^{irk \cdot x} J_{s+r}(\Omega_1, v, \Omega_2) P_s(m, \mathcal{M}) \bar{J}_s(\Omega_1, v, \Omega_2). \quad (66)$$

To link this with our perturbative results, it is instructive to consider the  $r = -1$  terms in this double sum with  $s$  ranging from  $-1$  to  $2$ . Expanding the Bessel functions (61) in terms of the In and Out representations, (59) and (60), gives for this part of (66) the explicit sum:

$$\begin{aligned} e^{-ik \cdot x} & \left( \left( \frac{1}{2} I^2 - \frac{1}{2} v \right) P_{-1}(m, \mathcal{M}) O \right. \\ & - O \left( \frac{1}{2} I^2 - \frac{1}{2} v \right) P_0(m, \mathcal{M}) + IP_0(m, \mathcal{M})(1 - IO) \\ & + P_1(m, \mathcal{M}) O \left( \frac{1}{2} I^2 + \frac{1}{2} v \right) - (1 - IO) P_1(m, \mathcal{M}) I \\ & \left. - OP_2(m, \mathcal{M}) \left( \frac{1}{2} I^2 + \frac{1}{2} v \right) \right). \end{aligned} \quad (67)$$

These terms are precisely the sum of the sidebands derived in (6) and the tree level part of (42), both with  $n = 0$ .

This identification of the momentum dependence in (67) with perturbative structures is gratifying and hints at the underlying logic of how to group the perturbative terms together.

In the perturbative formulation developed here we have not yet incorporated the fact that the laser background breaks translational invariance. This means that our momentum space description, where we have presented a direct way to calculate loop corrections for the sidebands, requires modification.

From the exact solution (56) we see that the modification is very simple. In addition to the standard momentum space factor  $e^{-ip \cdot (x-y)}$  which multiplies (56), we see an  $e^{irk \cdot x}$  factor which explicitly violates translation invariance. This, though, can be exploited to organize the perturbative discussion and will allow us to group terms consistently.

The key observation to note is that all the terms in (67) share a common homogeneity in the absorption and emission fields. Indeed, they all include either an absorption and no emissions, or two absorptions and one emission. From this simple observation it follows that if we multiply each absorption term by a factor of  $e^{-ik \cdot x}$  and each emission term by  $e^{ik \cdot x}$ , then we obtain the overall phase factor seen in (67). In a similar way, the terms in

$$\text{Diagram} = \sum_r \sum_s J_{s+r}(I: v: O) P_s(m, \mathcal{M}) \bar{J}_s(I: v: O)$$

FIG. 17. Strong field one-loop conjecture.

Fig. 10, say, would be accompanied by a factor of  $e^{-i2k \cdot x}$ , while Fig. 12 would pick up a factor of  $e^{i2k \cdot x}$ .

This motivates the following redefinition of the fundamental absorption vertex (2) by including the exponential factor:

$$A = -i\mathcal{A} \rightarrow -ie^{-ik \cdot x}\mathcal{A}. \quad (68)$$

Similarly, we have the associated dual redefinition  $E := -\bar{A} \rightarrow -ie^{ik \cdot x}\bar{A}^*$ . Hence from (26) we see that  $v \rightarrow e^{-i2k \cdot x}v$  while, from (31),  $v^* \rightarrow e^{i2k \cdot x}v^*$ . In terms of these redefinitions,  $v$  and  $v^*$  are not the same, so the notational convenience used earlier now becomes a genuine distinction. It is also clear now how to combine the perturbative terms in a physically correct manner in terms of commensurate powers of absorption minus emission. Note that the mass term  $\mathcal{M}$  picks up no spatial dependence under this redefinition.

Armed with this reformulation of the theory, we now conjecture how the double line description Fig. 1 and its one-loop corrections symbolized by Fig. 2 should be defined in terms of the renormalized fields (47)–(49), within the minimal subtraction scheme defined by (54) and (55). Our conjecture is that, in terms of the renormalized masses introduced in this paper, we have the identification summarized in Fig. 17, where we have introduced the condensed notation that  $J_s(I: v: O) = J_s(-(I + O), v, -i(I - O))$ .

This result holds at the tree level to all orders, and we have seen in this paper that, at one loop in Feynman gauge, it also holds for the ultraviolet poles in several different sidebands.

## VII. CONCLUSIONS

In this paper we have developed a perturbative description of the propagation of an electron in a plane wave background. There are two expansions here: one in the interactions with the background and an expansion up to one loop in perturbation theory. Each interaction with the background generates sideband structures. We have seen that the loop corrections maintain these sidebands. This means that a multiplicative renormalization of the theory

can be carried out in this formulation. We have worked in Feynman gauge and the background chosen was the full elliptic class of polarizations to bring out any polarization dependence.

The tree level sideband approach to charge propagation in a laser has the advantage that, at its heart, it identifies with each sideband a standard propagator, with momentum shifted by some multiple of the background momentum. These propagators are then multiplied by well-defined terms characterizing the laser. We have carried out a weak field expansion and explicitly calculated leading contributions to various sidebands.

Using dimensional regularization, we have calculated the one-loop, ultraviolet divergent poles in these sideband structures. They included multiple absorptions, multiple emissions, and, importantly, contributions from a mixture of absorptions and emissions. These last structures are responsible for the background induced electron mass shift.

Our calculations have revealed that the loop corrections to the sidebands replace the propagators by their equivalent standard one-loop corrections. This is a minimal requirement for multiplicative renormalization. However, we also found one additional ultraviolet divergent correction. This pole is a correction to the laser induced mass shift. As this was unexpected, we have verified that the same correction occurs in different sidebands in the Volkov propagator.

We have seen how to renormalize these divergences in terms of the usual one-loop renormalization without a background, plus an additional multiplicative renormalization of the laser induced mass shift. Inspired by the all orders tree level description, we have been able to conjecture an all orders expression for the full one-loop corrections in this class of backgrounds.

To complete this conjecture for the pole structure requires a proof that it holds to all orders in emissions and absorptions from the laser. A stronger form of the conjecture involves showing that the ultraviolet finite loop corrections, and any infrared divergences [25], are also compatible with this structure. Finally, it is important to study these results in other gauges.

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